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A CHARACTERIZATION OF NON-DUNFORD–PETTIS OPERATORS ON L^1

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ABSTRACT

It is shown that if $T: L^1 \to L^1$ is a bounded linear operator failing the Dunford–Pettis property, then T fixes a copy of $\bigoplus_{l^1}(l^2)$.

Introduction

If X and Y are Banach spaces, then a bounded linear operator $T: X \to Y$ is called a Dunford–Pettis operator provided T maps weakly compact sets onto norm compact sets.

We denote L^1 the Banach space of Lebesgue integrable function (classes) on $[0, 1]$.

An operator $T: L^1 \to Y$ is Dunford–Pettis if and only if Ti_∞ is a compact operator, where $i_\infty: L^\infty \to L^1$ is the canonical injection (see [2] or [1], prop. 1). In fact, the following result holds (see [1], prop. 5).

LEMMA 1. *If $T: L^1 \to Y$ fails the Dunford–Pettis property, then there exist a measurable subset Ω of $[0, 1]$, a weakly null sequence (φ_n) in L^1 and $\varepsilon > 0$ satisfying*

(1) $m(\Omega) > 0$,

(2) $\|\varphi_n\|_\infty \leqq 1$,

(3) $\overline{\lim}_r \|T(f\varphi_r)\| \geqq \varepsilon \|f\|_1$ *if* $f \in L^1(\Omega)$ *and* $f \geqq 0$.

(We denote $L^1(\Omega)$ the members of L^1 vanishing outside Ω.)

It was shown by H. P. Rosenthal (see [2], th. 1) that if $T: L^1 \to L^1$ is non-Dunford–Pettis, then thehere exists a subspace Z of L^1 such that Z is isomorphic to l^2 and the restriction $T \mid Z$ is an isomorphism. We will use here the following more quantitative version of this fact.

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LEMMA 2. *For each $\varepsilon > 0$, there exists a constant $C_\varepsilon < \infty$ such that if $T: L^1 \to L^1$ is an operator of $\|T\| \leqq 1$ and (φ_r) a weakly null sequence in L^1 so that $\|\varphi_r\|_\infty \leqq 1$ and $\|T(\varphi_r)\|_1 > \varepsilon$ for each r, then there is a sequence (Ψ_r) in L^1 satisfying*

$$\frac{1}{C_\varepsilon}\left(\sum_r |a_r|^2\right)^{1/2} \leqq \left\|\sum_r a_r T(\Psi_r)\right\|_1 \leqq \left\|\sum_r a_r \Psi_r\right\|_1 \leqq C_\varepsilon \left(\sum_r |a_r|^2\right)^{1/2}$$

for all scalars (a_r).

Denote $\bigoplus_{l^1}(l^2) = (l^2 \oplus l^2 \oplus \cdots)_{l^1}$ the l^1-sum of spaces l^2.

In this paper, previously mentioned results will be improved in the following way.

THEOREM. *For any non-Dunford–Pettis operator $T: L^1 \to L^1$, there is a subspace Z of L^1, Z isomorphic to $\bigoplus_{l^1}(l^2)$, such that $T \mid Z$ is an isomorphism.*

This solves affirmatively a question raised in [2].

We say that a Banach space X has the Schur property provided weakly compact subsets of X are norm compact. As a consequence of the theorem, we get:

COROLLARY. *Any complemented subspace of L^1 failing the Schur property contains a copy of $\bigoplus_{l^1}(l^2)$.*

It is an open question whether or not non-Schur complemented subspaces of L^1 are isomorphic to L^1. By results of P. Enflo and T. Starbird (see [4]) a positive solution to this problem should be obtained provided these spaces always contain a copy of L^1.

Proof of the Theorem

Let us first remark that if (Z_n) is a sequence of disjointly supported subspaces of L^1 such that Z_n is isomorphic to l^2 with bounded isomorphism constant, then the subspace of L^1 generated by the Z_n is isomorphic to $\bigoplus_{l^1}(l^2)$.

If f is an L^1-function and A a measurable set, $f.A$ will denote the restriction of f to A. Denote m the Lebesgue-measure on $[0, 1]$. For A measurable with $m(A) > 0$, define the operators

$$R_A: L^1 \to L^1(A) \quad \text{and} \quad I_A: L^1(A) \to L^1$$

by

$$R_A(f) = f.A \quad \text{and} \quad I_A(f) = m(A)^{-1} f.$$

LEMMA 3. *Let $T: L^1 \to L^1$ be an operator and (φ_r) a weakly null sequence in L^1 satisfying*

(1) $\|\varphi_r\|_\infty \leqq 1$ *for each* r,

(2) *there exists $\rho > 0$ so that for any $\delta > 0$ there are sets A and S with $m(S) < \delta$ and $\overline{\lim}_r \|R_S T I_A(\varphi_r . A)\|_1 \geqq \rho$.*

Then T fixes a copy of $\bigoplus_{l^1}(l^2)$.

The main point in the proof is the application of Lemma 2 to the operators $R_S T I_A$ and the remark made at the beginning of this section. The details are completely standard and left to the reader.

So it remains to realize the conditions of Lemma 3.

Given a non-Dunford–Pettis operator $T: L^1 \to L^1$, let Ω, (φ_r) and ε be as in Lemma 1.

At this point, it is convenient to introduce some notation:

For sets A and S, we let

$$(A, S) = |R_S T I_A|,$$

the variation norm of the operator $R_S T I_A$, and

$$[A, S] = \overline{\lim_r} \|R_S T I_A(\varphi_r . A)\|_1.$$

The crucial part in the proof of the theorem is the following lemma:

LEMMA 4. *Suppose that A, S and $S' \subset S$ are given sets and let $\tau > 0$. Then there exist a subset A_1 of A and a subset S_1 of S satisfying*

(1) $S' \subset S_1$,

(2) $m(S_1 \setminus S') < \tau$,

(3) $(A_1, S_1) > (A, S') + \frac{1}{2}([A, S] - [A, S']) - \tau$.

PROOF. There is clearly a subsequence (φ_{r_i}) of (φ_r) so that

$$\|R_{S \setminus S'} T I_A(\varphi_{r_i} . A)\| > [A, S] - [A, S'] - \tau$$

holds for each i.

Passing then eventually to a further subsequence, it is routine to construct a sequence (ψ_i) in L^∞ satisfying

(i) $\|\psi_i\|_\infty \leqq 1$,

(ii) $\langle \psi_i, \psi_j \rangle = 0$ for $i \neq j$,

(iii) $\operatorname{supp} \psi_i \subset S \setminus S'$,

(iv) $\langle T I_A(\varphi_{r_i} . A), \psi_i \rangle > \frac{1}{2}([A, S] - [A, S'] - \tau)$.

Take $M > \|T\|/\tau$ and let $j > 4M^2/\tau^2$ be an integer.

Assume for $i = 1, \cdots, j$

$$\varphi_{r_i} . A = \sum_\alpha c_{i,\alpha} \chi_{B_\alpha}$$

where the $|c_{i,\alpha}| \leqq 1$ and (B_α) is a measurable partition of A with $m(B_\alpha) > 0$ for all α (this can be done by step function approximation). Then

$$|R_{S'} TI_A| = \sum_\alpha \frac{m(B_\alpha)}{m(A)} |R_{S'} TI_{B_\alpha}|$$

and for $i = 1, \cdots, j$

$$\langle TI_A(\varphi_{r_i} . A), \psi_i \rangle = \sum_\alpha c_{i,\alpha} \langle TI_A(\chi_{B_\alpha}), \psi_i \rangle$$

$$\leqq \sum_\alpha \frac{m(B_\alpha)}{m(A)} |\langle T(mB_\alpha)^{-1} \chi_{B_\alpha}), \psi_i \rangle|.$$

Since now

$$|R_{S'} TI_A| + \frac{1}{j} \sum_{i=1}^{j} \langle TI_A(\varphi_{r_i} . A), \psi_i \rangle > (A, S') + \frac{1}{2}([A, S] - [A, S'] - \tau)$$

there must be some α so that for $B = B_\alpha$

$$|R_{S'} TI_B| + \frac{1}{j} \sum_{i=1}^{j} |\langle T(m(B)^{-1} \chi_B), \psi_i \rangle| > (A, S') + \frac{1}{2}([A, S] - [A, S'] - \tau).$$

Take $A_1 = B \subset A$, $h = T(mB)^{-1} \chi_B)$, $h' = h . [|h| \leqq M]$ and $h'' = h . [|h| > M]$. Then

$$\frac{1}{j} \sum_{i=1}^{j} |\langle h, \psi_i \rangle| \leqq \frac{1}{j} \sum_{i=1}^{j} |\langle h', \psi_i \rangle| + \|h'' . (S \backslash S')\|_1.$$

Remark that by the Cauchy–Schwartz inequality

$$\frac{1}{j} \sum_{i=1}^{j} |\langle h', \psi_i \rangle| \leqq \|h'\|_2 \left\| \sum_{i=1}^{j} \frac{1}{j} \varepsilon_i \psi_i \right\|_2 \qquad (\varepsilon_i = \pm 1)$$

$$\leqq Mj^{-1/2} < \frac{\tau}{2}$$

and hence

$$\|h'' . (S \backslash S')\|_1 > \frac{1}{j} \sum_{i=1}^{j} |\langle h, \psi_i \rangle| - \frac{\tau}{2}.$$

If we define $S''=[|h|>M]\cap(S\backslash S')$, obviously

$$m(S'')\leqq M^{-1}\|h\|_1\leqq M^{-1}\|T\|<\tau.$$

The set $S_1=S'\cup S''$ satisfies (1) and (2). Also

$$\begin{aligned}|R_{S_1}TI_{A_1}|&=|R_{S'}TI_{A_1}|+|R_{S''}TI_{A_1}|\\&\geqq|R_{S'}TI_{A_1}|+\|h\,.\,S'\|_1\\&>|R_{S'}TI_{A_1}|+\frac{1}{j}\sum_{i=1}^{j}|\langle h,\psi_i\rangle|-\frac{\tau}{2}\\&>(A,S')+\frac{1}{2}([A,S]-[A,S'])-\tau\end{aligned}$$

as required.

Using Lemma 3, the proof of the theorem will be completed by showing the following

Lemma 5. *For any $\delta>0$, one can find a subset A of Ω with $m(A)>0$ and a set S such that $m(S)<\delta$ and $[A,S]>\varepsilon/2$.*

Proof. Denote U the unit interval. Reformulation of (3) in Lemma 1 yields that $[A,U]\geqq\varepsilon$ whenever A is a subset of Ω with $m(A)>0$. Assume the statement in the lemma wrong. Thus there is $\delta>0$ such that $[A,S]\leqq\varepsilon/2$ whenever $A\subset\Omega$, $m(A)>0$ and $S\subset U$, $m(S)<\delta$. Proceeding by induction, we construct a sequence (A_k) of subsets of Ω and a sequence (S_k) of subsets of U satisfying

(1) $A_{k+1}\subset A_k$,
(2) $S_{k+1}\supset S_k$,
(3) $m(A_k)>0$,
(4) $m(S_k)<\delta(1-2^{-k})$,
(5) $(A_{k+1},S_{k+1})>(A_k,S_k)+\frac{1}{2}([A_k,U]-[A_k,S_k])-2^{-k-1}\delta$.

Take $A_1=\Omega$ and $S_1=\emptyset$. If now A_k and S_k are obtained, we apply Lemma 4 with $A=A_k$, $S=U$, $S'=S_k$ and $\tau=2^{-k-1}\delta$. This gives $A_{k+1}\subset A_k$ and $S_{k+1}\subset U$ such that $S_k\subset S_{k+1}$, $m(S_{k+1})\leqq m(S_k)+m(S_{k+1}\backslash S_k)<\delta(1-2^{-k})+\delta 2^{-k-1}=\delta(1-2^{-k-1})$ and (5).

Since $[A_k,U]\geqq\varepsilon$ and by hypothesis $[A_k,S_k]\leqq\varepsilon/2$, we have $[A_k,U]-[A_k,S_k]\geqq\varepsilon/2$ for each k. Iteration of (5) yields

$$\sup_k(A_k,S_k)\geqq\frac{1}{2}\sum_k([A_k,U]-[A_k,S_k])-\delta=\infty.$$

This contradicts however the fact that $(A,S)\leqq\|T\|$ for all sets A and S.

References

1. J. Bourgain, *Dunford–Pettis operators on L^1 and the Radon-Nikodym property*, Israel J. Math. **37** (1980), 34–47.

2. H. P. Rosenthal, *Convolution by a biased coin*, The Altgeld Book 1975/6, University of Illionis, Functional Analysis Seminar.

3. H. P. Rosenthal, *On subspaces L^p*, Ann. of Math. **97** (1973), 344–373.

4. T. Starbird, *Subspaces of L^1 containing L^1*, Ph. D. thesis, University of California, Berkeley, 1976.

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